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On the computation of the invariant vectors of a matrix.

Introduction.

Let P be a square matrix of order n with real or complex elements and let

(1) $\Delta(\lambda) = \det(\lambda I - P) = \lambda^n + k_1 \lambda^{n-1} + \ldots + k_n = 0$

be its characteristic equation with roots $\lambda_1, \ldots, \lambda_g$ ($\lambda_i \neq \lambda_k$ if $i \neq k$). The elementary divisors of $\lambda I - P$ are characterized by Segre's notation $(q_{1,1}; \ldots; q_{1,\nu_1}); \ldots; (q_{g,1}; \ldots; q_{g,\nu_g})$.

Here the $q_{h,j}$ are positive integers with

(2) $$q_{h,1} \geqq q_{h,2} \geqq \cdots \geqq q_{h,\nu_h}$$

$(h = 1,2,\ldots,g)$ and

$$\sum_{h=1}^{g} \sum_{j=1}^{\nu_h} q_{h,j} = n.$$

The above notation denotes that for $0 \leqq k \leqq n-1$ the k-th minors of $\lambda I - P$ (which are of order n-k) have the common divisor

$$\alpha_k(\lambda) = \prod_{h=1}^{g} (\lambda - \lambda_h)^{\varepsilon_{h,k}}$$

where

$$\varepsilon_{h,k} = 0 \qquad \text{if } k \geqq \nu_h$$
$$= q_{h,k+1} + \ldots + q_{h,\nu_h} \qquad \text{if } k < \nu_h;$$

(the factors $\alpha_k(\lambda)$ resp. $(\lambda - \lambda_h)^{\varepsilon_{h,k}}$ are called the elementary factors resp, the elementary divisors). Especially, putting

(3) $$q_h = \sum_{j=1}^{\nu_h} q_{h,j}$$

$(q_1+q_2+\ldots+q_g = n)$, we have

$$\Delta(\lambda) = \prod_{h=1}^{g} (\lambda - \lambda_h)^{q_h}.$$

Hence λ_h is an eigenvalue of P of multiplicity q_h. Further, $\lambda_h I-P$ is of rank $n-\nu_h$ whence it follows that there are exactly ν_h linearly independent eigenvectors of P belonging to the eigenvalue λ_h. By (3) and $q_{h,j} \geqq 1$ we have $\nu_h \leqq q_h$.

It is known that there can be found a set of n linearly independent vectors

$$e_{h,j,k} \qquad (h = 1,\ldots,g;\ j = 1,\ldots,\nu_h;\ k = 1,\ldots,q_{h,j})$$

which satisfies the system of n equations

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(4) $$(P-\lambda_h I)\, e_{h,j,k} = e_{h,j,k-1},$$

provided that we put $e_{h,j,o} = 0$. These vectors $e_{h,j,k}$ are called <u>invariant vectors</u>. The ν_h invariant vectors $e_{h,j,1}$ ($j = 1, \ldots, \nu_h$) are the ν_h eigenvectors belonging to the eigenvalue λ_h.

Now, we form a non-singular matrix T of order n whose column vectors are the invariant vectors $e_{h,j,k}$. These vectors are supposed to be ordered in such a way that $e_{h,j,k}$ precedes $e_{h',j',k'}$ if

$$h < h'$$
$$\text{or } h = h' \text{ and } j < j'$$
$$\text{or } h = h',\ j = j' \text{ and } k < k'.$$

Further let $P_{h,j}$ ($h=1,\ldots,g$; $j=1,\ldots,\nu_h$) be the matrix of order $q_{h,j}$ defined by

$$P_{h,j} = \left\| \begin{matrix} \lambda_h & 1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_h & 1 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & \lambda_h & 1 \\ 0 & 0 & 0 & \ldots & 0 & \lambda_h \end{matrix} \right\|$$

(all elements of the main diagonal equal λ_h; all elements of the parallel adjacent to the eight equal 1; all remaining elements are zero).

Finally, we introduce the following matrix $^*P$ of order n

$$^*P = \left\| \begin{matrix} P_{1,1} & & & & & \\ & \ddots\, P_{1,\nu_1} & & & & \\ & & P_{2,1} & & & \\ & & & \ddots\, P_{2,\nu_2} & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & P_{g,1} & \\ & & & & & \ddots\, P_{g,\nu_g} \end{matrix} \right\|$$

composed of blocks $P_{h,j}$ arranged so that the main diagonal of each lies on the main diagonal of $^*P$; the other coordinates being 0. This matrix $^*P$ has a so-called canonical form. With any matrix A there corresponds at least one non-singular matrix S for which $^*A = S^{-1}AS$ has a canonical form; (one says that $^*A$ is a canonical form of A). This canonical form is unique save for the order in which the blocks along the main diagonal of $^*A$ are arranged.

Using the above matrices T and $^*P$ the system of n equalities (4) can be replaced by

(5) $$PT = T^*P \quad \text{or} \quad ^*P = T^{-1}PT.$$

Hence the determination of a set of n linearly independent invariant vectors (which satisfy (4)) is equivalent to the determination of the non-singular matrix T which transforms P into its canonical form $^{*}P = T^{-1}PT$.

Summary.

Let P be a given square matrix of order n with real or complex elements. In this report an exposition will be given of a new method for the computation of the eigenvalues, the invariant vectors, the Segre numbers etc. of P. This method is a generalization of a known method for the computation of eigenvectors (cf. the bibliography at the end).

The method.

In this parapgraph the announced method will be explained. All unproved assertions will be demonstrated in the last section.

First of all, one successively computes

$$(6)\quad \begin{cases} B_1 = P + k_1 I & \text{where} \quad k_1 = -\operatorname{trace} P \\ B_2 = PB_1 + k_2 I & \text{"} \quad k_2 = -\tfrac{1}{2}\operatorname{trace} PB_1 \\ \cdots & \\ B_{n-1} = PB_{n-2} + k_{n-1} I & \text{"} \quad k_{n-1} = -\tfrac{1}{n-1}\operatorname{trace} PB_{n-2} \\ B_n = PB_{n-1} + k_n I & \text{"} \quad k_n = -\tfrac{1}{n}\operatorname{trace} PB_{n-1}. \end{cases}$$

The equality $B_n = 0$ yields a check (i.e. PB_{n-1} is a diagonal matrix).

Next, one computes the different roots $\lambda_1, \ldots, \lambda_g$ of the equation

$$\Delta(\lambda) = \lambda^n + k_1 \lambda^{n-1} + \ldots + k_n = 0$$

and the multiplicity q_h of the root λ_h ($h = 1, \ldots, g$); we have $q_1 + q_2 + \ldots + q_g = n$. These roots $\lambda_1, \ldots, \lambda_g$ are the eigenvalues of P.

In order to compute a set of q_1 linearly independent invariant vectors belonging to the eigenvalue λ_1 we introduce

$$(7)\quad Q(\lambda) = I\lambda^{n-1} + B_1\lambda^{n-2} + B_2\lambda^{n-3} + \ldots + B_{n-1}.$$

This matrix satisfies*)

$$(8)\quad (\lambda I - P)\ Q(\lambda) = \Delta(\lambda).$$

Hence, by $\Delta^{(h)}(\lambda_1) = 0$ $(h = 0, 1, \ldots, q_1 - 1)$, it follows that the matrices

$$(9)\quad Q_h = \frac{1}{h!}\left[\frac{d^h}{d\lambda^h}\, Q(\lambda)\right]_{\lambda=\lambda_1} \qquad (Q_{-1} = 0)$$

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*) Especially, it follows from (8), that $P^{-1} = -\frac{1}{\Delta(0)}\, Q(0) = \frac{-1}{\Delta(0)} B_{n-1}$.

satisfy

(10) $$(P - \lambda_1)\ Q_h = Q_{h-1} \quad (h = 0,1,\dots,q_1-1).$$

Let $q_{1,j}$ $(j = 1,\dots,\nu_1)$ be the Segre numbres of P belonging to the eigenvalue λ_1 (see introduction). By a_k we denote the number of the $q_{1,j}$ which are equal to k ($k = 1,2,\dots,q_1$). If these numbers a_k are known, then, also the Segre numbers $q_{1,j}$.

We observe that

(11) $$q_1 = \sum_{j=1}^{\nu_1} q_{1,j} = \sum_{k=1}^{q_1} k a_k .$$

Finally, we assert thath the rank r_h of Q_h is given by

(12) $$r_h = a_{q_1-h} + 2a_{q_1-h+1} + 3a_{q_1-h+2} + \dots + (h+1)a_{q_1}$$
$$(h = 0,1,2,\dots,q_1-1).$$

The above properties enable us to compute a set of q_1 linearly independent invariant vectors belonging to λ_1. Such a set consists of vectors

$$f_{k,\ell,m} \quad (k = 1,\dots,q_1; \quad \ell = 1,\dots,a_k; \quad m = 1,\dots,k)$$

(omitting those k for which $a_k = 0$) which satisfy

$$(P - \lambda_1)\ f_{k,\ell,m} = f_{k,\ell,m-1}$$

provided that we put $f_{k,\ell,o} = 0$. The $\nu_1 = \sum a_k$ vectors $f_{k,\ell,1}$ are the eigenvectors belonging to λ_1.

The computation runs as follows. First of all, one computes the matrices Q_h $(h = 0,1,\dots,q_1-1)$ from (7) and (9). Afterwards, one determines the rank r_h of Q_h $(h = 0,1,\dots,q_1-1)$. By (12) we have

(13) $$a_{q_1-h} = r_h - 2r_{h-1} + r_{h-2} \quad (h = 0,\dots,q_1)$$

provided that we put $r_{-1} = r_{-2} = 0$. Thus the numbers $a_1,\dots,a_{q_1}$ are known and, hence, also the Segre numbers belonging to λ_1.

It suffices to determine $\nu_1 = \sum_{k=1}^{q_1} a_k$ eigenvectors belonging to λ_1 in such a way that (for $k = 1,\dots,q_1$; $a_k > 0$) exactly a_k eigenvectors $f_{k,\ell,1}$ ($\ell = 1,\dots,a_k$) are obtained as a proper linear composition of the column vectors of Q_{q_1-k} and that the resulting $\nu_1 = \sum a_k$ eigenvectors are linearly independent. If this has been done we are ready. For the $f_{k,\ell,m}$ $(m = 2,\dots,k)$ can now be obtained by replacing in the linear combination of the columns of Q_{q_1-k}, which constitutes $f_{k,\ell,1}$, any column by the corresponding column of $Q_{q_1-k+m-1}$. By (10) these vectors $f_{k,\ell,m}$ satisfy

(15) $$(P - \lambda_1)\ f_{k,\ell,m} = f_{k,\ell,m-1}$$

$(m = 2,3,\dots,k)$. Further, $f_{k,\ell,1}$ is an eigenvector.

Hence (15) holds if $1 \leqq k \leqq q_1$, $a_k > 0$, $1 \leqq \ell \leqq a_k$ and $1 \leqq m \leqq k$

provided that we put $f_{k,\ell,o} = 0$. Using (15) and the linear indenpendency of the ν_1 eigenvectors $f_{k,\ell,1}$, one easily proves that the $q_1 = \sum k a_k$ vectors $f_{k,\ell,m}$ are linear independent. Thus, the required set of q_1 invariant vectors has been found.

There remains the above mentioned computation of $\nu_1 = \sum a_k$ linearly independent eigenvectors $f_{k,\ell,1}$ ($k = 1,2,\ldots,q_1$; $a_k > 0$; $\ell = 1,2,\ldots,a_k$) in such a way that $f_{k,\ell,1}$ is linear dependent on the columns of Q_{q_1-k}.

From (12) we obtain

$$0 \leqq r_1 \leqq r_2 \leqq \cdots \leqq r_{q_1-1} = q_1 > 0.$$

Hence the integer $\rho = q_1 - p$ $(0 \leqq \rho \leqq q_1 - 1)$ can be chosen in a uniquely determined manner in such a way that $r_h = 0$ for $h < \rho$, whereas $r_\rho > 0$. By (12) we have $a_h = 0$ if $h > p$ (i.e. all $q_{1,j}$ are at most equal to p) and $a_p = r_\rho$ (i.e. there are exactly r_ρ among the $q_{1,j}$ which are equal to p).

From (10) and $Q_{\rho-1} = 0$ (for $r_{\rho-1} = 0$) we conclude that any column of Q_ρ is an eigenvector of P belonging to λ_1. Let

$$f_{p,\ell,1} \qquad (\ell = 1,\ldots,a_p)$$

be a set of a_p independent column vectors of Q_ρ (which is of rank $a_p = r_\rho$).

If $pa_p = q_1$ we are ready; (then $a_h = 0$ for $h \neq p$). Otherwise, by (11), there exists an integer with $1 \leqq s < p$ and $a_s > 0$; let s be the largest integer with this property. By (12), the matrix Q_{q_1-s-1} is of rank $(p-s)a_p$ whereas Q_{q_1-s} has the rank $(a_s+a_p) + (p-s)a_p$. Let

$$x_h \qquad (h = 1,2,\ldots,(p-s)a_p)$$

be a set of $(p-s)a_p$ linearly independent column vectors of Q_{q_1-s-1}. Let y_h be the column of Q_{q_1-s} which corresponds to x_h. Further, let z_h $(h = 1,\ldots,a_s+a_p)$ be a set of a_s+a_p column vectors of Q_{q_1-s} which together with the $(p-s)a_p$ (linearly independent) vectors y_h form a set of $(a_s+a_p) + (p-s)a_p$ linearly independent vectors.

From (10) we infer

$$(P - \lambda_1)y_h = x_h \qquad (h = 1,\ldots,(p-s)a_p)$$

while $(P - \lambda_1)z_h$ is the column of Q_{q_1-s+1} which corresponds to z_h. This column is linearly dependent on the x_h. Hence

$$(P - \lambda_1)z_h = \alpha_{h,1}\, x_1 + \alpha_{h,2}\, x_2 + \ldots$$

where $\alpha_{h,k}$ is a scalar. Finally, we introduce the vector

$$\zeta_h = z_h - \alpha_{h,1}\, y_1 - \alpha_{h,2}\, y_2 - \ldots \qquad (h = 1,\ldots,a_s+a_p)$$

which satisfies

$$(P - \lambda_1)\zeta_h = 0 \qquad (h = 1,\ldots,a_s + a_p).$$

These $a_s + a_p$ vectors ζ_h are linearly independent. At least a_s vectors ζ_h

$$f_{s,1,1}, \ldots, f_{s,a_s,1} \qquad \text{(say)}$$

form together with

$$f_{p,1,1}, \ldots, f_{p,a_p,1}$$

a set of $a_s + a_p$ linearly independent eigenvectors of P.

If $pa_p + sa_s = q_1$ we are ready. Otherwise the procedure must be continued. The way of this procedure seems to be enough clarified by the above detailed exposition.

Example. Let be

$$P = \begin{Vmatrix} 12 & -2 & -6 & -3 \\ 15 & -2 & -8 & -4 \\ -8 & 2 & 5 & 2 \\ 41 & -8 & -22 & -10 \end{Vmatrix}$$

From (6) we obtain, successively,

$$k_1 = -\text{ trace } P = -5$$

$$B_1 = P + k_1 I = \begin{Vmatrix} 7 & -2 & -6 & -3 \\ 15 & -7 & -8 & -4 \\ -8 & 2 & 0 & 2 \\ 41 & -8 & -22 & -15 \end{Vmatrix}$$

$$PB_1 = \begin{Vmatrix} -21 & 2 & 10 & 5 \\ -25 & 0 & 14 & 7 \\ 16 & -4 & -12 & -4 \\ -67 & 10 & 38 & 15 \end{Vmatrix}$$

$$k_2 = -\tfrac{1}{2} \text{ trace } PB_1 = +9$$

$$B_2 = PB_1 + k_2 I = \begin{Vmatrix} -12 & 2 & 10 & 5 \\ -25 & 9 & 14 & 7 \\ 16 & -4 & -3 & -4 \\ -67 & 10 & 38 & 24 \end{Vmatrix}$$

$$PB_2 = \begin{Vmatrix} 11 & 0 & -4 & -2 \\ 10 & 4 & -6 & -3 \\ -8 & 2 & 9 & 2 \\ 26 & -2 & -16 & -3 \end{Vmatrix}$$

$$k_3 = -1/3 \text{ trace } PB_2 = -7$$

$$B_3 = PB_2 + k_3 I = \begin{Vmatrix} 4 & 0 & -4 & -2 \\ 10 & -3 & -6 & -3 \\ -8 & 2 & 2 & 2 \\ 26 & -2 & -16 & -10 \end{Vmatrix}$$

$$PB_3 = \begin{Vmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{Vmatrix}$$

$$k_4 = -1/4 \text{ trace } PB_3 = +2$$

(by $PB_3 = -k_2 I$, it suffices to compute only one diagonal element of PB_3). Thus, we have*)

$$Q(\lambda) = \lambda^3 + \lambda^2 B_1 + \lambda B_2 + B_3$$

and

$$\Delta(\lambda) = \lambda^4 - 5\lambda^3 + 9\lambda^2 - 7\lambda + 2$$
$$= (\lambda - 2)(\lambda - 1)^3.$$

Hence, $\lambda_1 = 2$ is a simple eigenvalue while $\lambda_2 = 1$ is an eigenvalue of multiplicity 3. Any column vector of

$$Q(2) = 8I + 4B_1 + 2B_2 + B_3 = \begin{Vmatrix} 16 & -4 & -8 & -4 \\ 20 & -5 & -10 & -5 \\ -8 & 2 & 4 & 2 \\ 56 & -14 & -28 & -14 \end{Vmatrix}$$

is, save for multiplication with a scalar, equal to the only invariant vector (= eigenvector) x_1 belonging to $\lambda_1 = 2$. We take

$$x_1 = \{-4, -5, +2, -14\};$$

(it suffices to compute only one non-zero column of $Q(2)$).

In order to determine the Segre numbers belonging to $\lambda_2 = 3$ it suffices to know the rank r_1 of Q_1, where

$$Q_h = \frac{1}{h!}\left[\frac{d^h}{d\lambda^h} Q(\lambda)\right]_{\lambda=1} \qquad (h = 0,1,2).$$

For we have from (12) for the rank r_h of Q_h

Segre's notation	r_0	r_1	r_2
(3)	1	2	3
(21)	0	1	3
(111)	0	0	3

In the above special case, we have

$$Q_1 = 3 + 2B_1 + B_2 = \begin{Vmatrix} 5 & -2 & -2 & -1 \\ 5 & -2 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 15 & -6 & -6 & -3 \end{Vmatrix}.$$

Hence $r_1 = 1$, which implies the Segre numbers $q_{2,1} = 2$ and $q_{2,2} = 1$. There are exactly two eigenvectors belonging to $\lambda_2 = 1$. One can be obtained as an arbitrary non-zero column vector of Q_1.

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*) By (8) we have $P^{-1} = -\frac{1}{\Delta(o)} Q(o) = -\frac{1}{2}B_3$.

We choose

$$x_2 = [-1,\ -1,\ 0,\ -3]$$

Finally, we have

$$Q_2 = 3I + B_1 = (k_1+3)I + P = -\ 2I + P = \left\| \begin{matrix} 10 & -2 & -6 & -3 \\ 15 & -4 & -8 & -4 \\ -8 & 2 & 3 & 2 \\ 41 & -8 & -22 & -12 \end{matrix} \right\|$$

Let y_i $(i = 1,2,3,4)$ be the i-th column vector of Q_2. By $(P-1)Q_2 = Q_1$ we have

$$(P - 1)y_2 = +\ 2x_2$$

$$(P - 1)y_4 = +\ x_2$$

Hence, the vectors

$$x_3 = y_4 = [-3,\ -4,\ 2,\ -12]$$

and

$$x_4 = y_2 - 2y_4 = [4,\ 4,\ -2,\ 16]$$

complete the set of invariant vactors. Here, x_4 is an eigenvector belonging to $\lambda_2 = 1$ which is linearly independent of the eigenvector x_2.

The non-singular matrix

$$T = (x_1;\ x_2;\ x_3;\ x_4) = \left\| \begin{matrix} -4 & -1 & -3 & 4 \\ -5 & -1 & -4 & 4 \\ 2 & 0 & 2 & -2 \\ -14 & -3 & -12 & 16 \end{matrix} \right\|$$

transforms the given matrix P into its canonical form

$$^{*}P = T^{-1}\ PT = \left\| \begin{matrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \right\| .$$

Theoretical investigations.

In this section we shall demonstrate the unproved assertions of the preceding paragraph. These are the following.

1. The coefficients k_m of the characteristic equation of P can be computed from (6).
2. The matrix $Q(\lambda)$ defined by (6) and (7) satisfies (8).
3. The rank r_h of Q_h (which is defined by (9)) satisfies (12) where a_h is equal to the number of Segre numbers $q_{1,j}$ (belonging to the eigenvalue λ_1) which are equal to h ($h = 1,2,\ldots,q_1$).

Remark.

Only for the proof of the third assertion we shall make use of the property that there exists a non-singular matrix T for which $^*P = T^{-1}PT$ has the canonical form. This assertion can also be proved without using the latter property. Then, the method of the preceding section yields, simultaneously, a proof of the reducibility of P into a canonical form.

Let P be a given square matrix of order n. If

(16) $$\Delta(\lambda) = \det(\lambda I - P) = \lambda^n + k_1\lambda^{n-1} + \ldots + k_n$$

the Hamilton-Cayley theorem states $\Delta(P) = 0$. Thus, we have, identically,

$$\Delta(\lambda)I = \Delta(\lambda I) - \Delta(P) = \sum_{i=0}^{n-1} k_i\left\{(\lambda I)^{n-i} - P^{n-i}\right\}$$

$$= (\lambda I - P)\sum_{i=0}^{n-1} k_i \sum_{m=i}^{n-1} \lambda^{n-m-1} P^{m-i}$$

$$= (\lambda I - P)\sum_{m=0}^{n-1} \lambda^{n-m-1} \sum_{i=0}^{m} k_i P^{m-i}$$

(where $k_o = 1$). Hence, putting

(17) $$B_m = P^m + k_1P^{m-1} + \ldots + k_mI \qquad (m = 0,1,\ldots,n)$$

($B_o = I$; $B_n = 0$) and

(18) $$Q(\lambda) = I\lambda^{n-1} + B_1\lambda^{n-2} + B_2\lambda^{n-3} + \ldots + B_{n-1},$$

we obtain

(19) $$(\lambda I - P)\,Q(\lambda) = \Delta(\lambda)I$$

i.e. the matrix $Q(\lambda)$ defined by (17) and (18) is the adjoint of $\lambda I - P$.

Now, we assert

(20) $$\operatorname{trace} Q(\lambda) = \frac{d\Delta(\lambda)}{d\lambda}$$

By continuity, we may restrict ourselves to the case where the n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all distinct and where $\lambda \neq \lambda_1$

$(i = 1,2,\ldots,n)$. We then have from (19)

$$\text{trace } Q(\lambda) = \Delta(\lambda) \text{ trace } (\lambda I - P)^{-1} = \sum_{i=1}^{n} \frac{\Delta(\lambda)}{\lambda - \lambda_i} = \frac{d\Delta(\lambda)}{d\lambda},$$

(for $(\lambda I - P)^{-1}$ has eigenvalues $(\lambda - \lambda_i)^{-1}$, which proves the assertion (20).

From (16), (18) and (20) we obtain

$$\text{trace } B_m = (n-m)k_m \qquad (m = 1,2,\ldots,n-1),$$

which holds also for $m = 0$ and $m = n$ by $k_o = 1, B_o = I$ and $B_n = 0$.

Hence, by (17)

$$\text{trace } PB_{m-1} = \text{trace } (B_m - k_m I) = (n-m)k_m - nk_m = -mk_m$$

or

(21) $$k_m = -\frac{1}{m} \text{ trace } PB_{m-1} \qquad (m = 1,\ldots,n)$$

Because (17) implies

$$B_m = PB_{m-1} + k_m I \qquad (m = 1,2,\ldots,n),$$

we have already proved the two first assertions mentioned at the beginning of this paragraph.

In order to prove the third assertion we remember that the non-singular matrix T of order n can be found in such a way that

$$^{*}P = T^{-1} PT$$

is the canonical form of P (see introduction) which has the same eigenvalues and the same Segre numbers. Introducing

(22) $$^{*}Q(\lambda) = T^{-1} Q(\lambda)T$$

we have from (8)

(23) $$(\lambda I - {}^{*}P)\, {}^{*}Q(\lambda) = \Delta(\lambda).$$

Hence, by $\Delta(\lambda) = \det(\lambda I - P) = \det(\lambda I - {}^{*}P)$, the matrix $^{*}Q(\lambda)$ is the adjoint of $^{*}P$. The matrix

$$Q_h = \frac{1}{h!}\left[\frac{d^h}{d\lambda^h}\, {}^{*}Q(\lambda)\right]_{\lambda = \lambda_1}$$

satisfies by (22)

$$^{*}Q_h = T^{-1} Q_h T$$

and, hence, has the same rank as Q_h, which is defined by (9).

Thus, it appears that, as to the proof of (12), we may restrict us to the case that P is a canonical form. Then, P is composed of blocks $P_{h,j}$ arranged so that the main diagonal of each lies on the main diagonal of P, while all remaining elements are zero. Let

$$q_{h,j} \qquad (j = 1,\ldots,\nu_h)$$

$(q_{h,1} \geqq q_{h,2} \geqq \cdots \geqq q_{h,\nu_h} \geqq 1)$ be the Segre numbers belonging to the eigenvalue λ_h $(h = 1,\ldots,g)$. There is a one-one correspondence between the blocks $P_{h,j}$ and the Segre numbers $q_{h,j}$; let

$q_{h,j}$ correspond to $P_{h,j}$. In this case, $P_{h,j}$ is the following square matrix of order $q_{h,j}$

$$P_{h,j} = \begin{Vmatrix} \lambda_h & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_h & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ \vdots & & & & 1 & 0 \\ \vdots & & & & \lambda_h & 1 \\ 0 & \cdots & & \cdots & 0 & \lambda_h \end{Vmatrix}$$

$Q(\lambda)$ is a continuous function of λ which for $\lambda \neq \lambda_h$ $(h = 1,\ldots,g)$ is equal to

$$\Delta(\lambda)\,(\lambda I - P)^{-1}$$

Here $(\lambda I - P)^{-1}$ is composed of blocks $(\lambda I - P_{h,j})^{-1}$ along the main diagonal. One easily establishes that, for $\lambda \neq \lambda_h$ $(h = 1,\ldots,g)$,

$$(\lambda I - P_{h,j})^{-1} = \begin{Vmatrix} (\lambda-\lambda_h)^{-1} & (\lambda-\lambda_h)^{-2} & \cdots & (\lambda-\lambda_h)^{-q_{h,j}} \\ 0 & (\lambda-\lambda_h)^{-1} & & \\ \vdots & & & \\ 0 & \cdots & \cdots & \cdots \end{Vmatrix}$$

where in each parallel of the main diagonal all elements are equal. Because of

$$\Delta(\lambda) = \prod_{h=1}^{g} (\lambda-\lambda_h)^{q_h}$$

we obtain that for any value λ the matrix $Q(\lambda)$ is composed of blocks of the form

$$R_{h,j}(\lambda) = \prod_{k\neq h} (\lambda-\lambda_k)^{q_k} \begin{Vmatrix} (\lambda-\lambda_h)^{q_h-1} & (\lambda-\lambda_h)^{q_h-2} & \cdots & (\lambda-\lambda_h)^{q_h-q_{h,j}} \\ 0 & (\lambda-\lambda_h)^{q_h-1} & & \\ \vdots & & & \\ 0 & \cdots & \cdots & \cdots \end{Vmatrix}$$

where, again, all elements of an arbitrary parallel of the main diagonal are equal. Let $r_{h,j,m}$ be the rank of the matrix

$$Q_{h,j,m} = \frac{1}{m!}\left[\frac{d^m}{d\lambda^m} R_{h,j}(\lambda)\right]_{\lambda=\lambda_1} \qquad (0 \leqq m \leqq q_1-1).$$

We then have

$$\begin{aligned} r_{h,j,m} &= 0 \quad \text{if} \quad h \neq 1 \\ &= 0 \quad \text{if} \quad h = 1 \text{ and } m+1-(q_1-q_{1,j}) \leqq 0 \\ &= i \quad \text{if} \quad h = 1 \text{ and } m+1-(q_1-q_{1,j}) = i \\ &\qquad\qquad \text{with } 1 \leqq i \leqq q_{1,j}. \end{aligned}$$

Now, the matrix Q_m defined by

$$Q_m = \frac{1}{m!}\left[\frac{d^m}{d\lambda^m} \; Q(\lambda)\right]_{\lambda=\lambda_1} \qquad (0 \leqq m \leqq q_1-1)$$

is composed of blocks $Q_{h,j,m}$ $(h = 1,\ldots,g;\ j = 1,\ldots,\nu_h)$ and , hence, has the rank

$$r_m = \sum_{h,j} r_{h,j,m} = \sum_{j=1}^{\nu_1} r_{1,j.m} =$$

$$= \sum_{q_{1,j}=q_1-m} 1 + 2 \sum_{q_{1,j}=q_1-m+1} 1 + \ldots + (m+1) \sum_{q_{1,j}=q_1} 1$$

$$= a_{q_1-m} + 2\, a_{q_1-m+1} + \ldots + (m+1)a_{q_1}$$

if a_k $(k = 1,\ldots,q_1)$ denotes the number of the $q_{1,j}$ which are equal to k. This proves formula (12).

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